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ROCKET BOOSTER CONTROL

SECTION 6

TIME-OPTIMAL BOUNDED

PHASE COORDINATE CONTROL

OF LINEAR RECURRENCE SYSTEMS

NASA Contract NASw-563

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ROCKET BOOSTER CONTROL,

SECTION 6

TIME-OPTIMAL BOUNDED PHASE COORDINATE CONTROL OF LINEAR RECURRENCE SYSTEMS

(NASA Contract NASw-563)

Prepared by: E B Lee
E. B. Lee 26 Dec. 1963 12p refs
Research Consultant

Supervised by: C. R. Stone
C. R. Stone
Research Supervisor

Approved by: O H Schuck
O. H. Schuck
Director
MPG Research

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Minneapolis-HONEYWELL Regulator Co., Minn.
MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT
Minneapolis, Minnesota

FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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TIME OPTIMAL BOUNDED PHASE
COORDINATE CONTROL OF LINEAR
RECURRENCE SYSTEMS*

by E. B. Lee†

ABSTRACT

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The time optimal control problem with bounded phase coordinates is considered for systems modeled by linear recurrence equations. It is shown that linear inequality constraints of the phase coordinates can be transformed into similar linear inequality constraints on the control variables. Methods for finding the so-constrained minimum are discussed. A convergent computational scheme is presented, which, unfortunately, involves a large amount of equipment for implementation. It does not seem practical to solve this problem on line using present computer technology.

Author

ANALYSIS

The real recurrence equation

$$x(r+1) = A(r)\ x(r) + \hat{B}(r)\ u(r) \tag{1}$$

will be considered, where $x(r)$, an n vector, is the system state; $u(r)$, an m vector, is the control; $A(r)$ and $\hat{B}(r)$ are m x n and n x m bounded matrices respectively; and $r = 0,1,2,\ldots$ denotes the stage of the evolution. It is assumed that $\det A(r) \neq 0$ for $r = 0,1,2,\ldots$

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* Prepared under contract NASw-563 for the NASA

† Research Consultant, Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota

The problem studied is how to select the sequence of points $[u(0), u(1), \ldots] = u$ from $\Omega : |u^j| \leq 1;\ j = 1,2,\ldots,m$ so that $x(r)$, the response of the above recurrence equation, when subjected to the sequence u, moves from $x(0) = x_o$ to an intersection with a prescribed target G in the minimum number of stages r and at each stage of the response $|x^k(r)| \leq B^k;\ k = 1,2,\ldots,n$, where B^k are constants, (linear combinations of the x^k's are similarly treated.)

If $W(r+1) = A(r)\ W(r)$ for $r = 0,1,2,\ldots$ with $W(0) = I$ then

$$x(r) = W(r)\ x_o + W(r) \sum_{j=1}^{r} W^{-1}(j)\ \hat{B}(j-1)\ u(j-1) \qquad (2)$$

The constraint: $|x^k(r)| \leq B^k$, $k = 1,2,\ldots n$ will now be transformed into a corresponding inequality constraint involving the control sequence $u = [u(0), u(1) \ldots u(T-1)]$ for fixed integer $T > 0$. Denote the closed convex set $\Omega \times \Omega \times \ldots \times \Omega \subset R^{mT}$ by $\hat{\Omega}$. A control sequence $u = [u(0), u(1) \ldots u(T-1)]$ is allowable if it belongs to $\hat{\Omega}$.

For $r = 0$, $x(0) = W(0)\ x_o = x_o$, which does not depend on the control sequence but demands that the initial data is such that $|x^k(0)| \leq B^k;\ k = 1,2,\ldots n$. For $r = 1$,

$$\begin{aligned} |x^k(1)| &= |W^k_j(1)\ x^j(0) + W^k_j(1)\ W^{-1j}_{\ell}(1)\ B^{\ell}_{\nu}(0)\ u^{\nu}(0)| \\ &\leq B^k;\ k = 1,2,\ldots n. \end{aligned}$$

Or

$$\left| \sum_{j=1}^{n} W^k_j(1)\ x^j(1)\ x^j(0) + \sum_{\ell=1}^{m} \gamma^k_{\ell}\ u(0) \right| \leq B^k;\ k = 1,2,\ldots,n$$

where $W^k_j(1)$ denotes the kj^{th} element of the matrix $W(1)$ and repeated index denotes summation with respect to that index. The above inequality on $u(0)$ defines a closed convex subset in the space R^{mT} of variables $u = [u(0), u(1) \ldots u(T-1]$. In fact, this subset is defined by hyperplanes. This subset of R^{mT} is denoted by $\Lambda(0)$.

For $r = 2$,

$$|x^k(s)| = |W^k_j(2)\ x^j(0) + W^k_j(2)\ W^{-1}_\ell(1)\ B_\nu(0)\ u^\nu(0)$$

$$+ W^k_j(2)\ W^{-1j}_\ell(2)\ B^\ell_\nu(1)\ u^\nu(1)| \leq B^k,$$

The so defined subset of R^{mT} is denoted by $\Lambda(1)$.

The subsets

$$\Lambda(2), \Lambda(3), \ldots, \Lambda(T-1)$$

of R^{mT} are similarly defined.

$$\Gamma(T) = \hat{\Omega} \cap \Lambda(0) \cap \ldots \cap \Lambda(T-1) \subset R^{mT}$$

is closed and convex (maybe empty). If a control sequence $u = [u(0) \ldots u(T-1)]$ belongs to the nonempty set $\Gamma(T)$, the corresponding response is such that $|x^k(t)| \leq B^k$ for $r = 0, 1, 2, \ldots,$ and $k = 1, 2, \ldots n$. Conversely, if $|x^k(r)| \leq B^k$; $r = 0, 1, 2, \ldots T$; $k = 1, 2, \ldots, n$ with $u \in \Omega$ the corresponding control sequence u belongs to $\Gamma(T)$. Thus the problem of optimum control with bounded phase coordinates can be considered as one of finding the best control sequence u in some closed convex constraint set $\Gamma(T) \subset R^{mT}$.

As has been done for the case where the phase variables are not bounded the target $G = \{x | x\ H' x \leq c\}$, where $c > 0$ is a constant and $H = H' > 0$. Introducing the positive definite function

$V(x) = x \cdot Hx$ and the error function $E(x) = x \cdot Hx - c$, the time optimal control problem with bounded phase coordinates is that of finding the first $T > 0$ such that $E(x) = 0$ for some $x(T)$ belonging to the set of attainability $K(T,x_o)$. $K(T,x_o)$ is the collection of end points of responses $x(T)$ which initiate at x_o for all control sequences $u = [u(0), u(1),\ldots u(T-1)]$ belonging to $\Gamma(T)$. Since the function which maps a point $u \in \Gamma(T) \subset R^{mT}$ into a point $x(T) \in R^n$ is linear, the image of the closed convex set $\Gamma(T)$ is a closed convex set, $K(T,x_o)$. Thus if $\Gamma(T)$ is not empty for fixed $T < T^*$ there is a unique point x^* of $K(T,x_o)$ where $V(x)$ is a minimum (here T^* denotes the first (smallest) T for which $E(x) = 0$ with $x \in K(T,x_o)$). The method of finding an optimum control is to increase T one step at a time finding at each time the point x^* of $K(T,x_o)$ which minimizes $E(x)$. When a T, namely T^*, is found for which $E(x) = 0$ for some $x \in K(T^*,x_o)$ an optimum control sequence is known. An attempt is now made to find a path $u(t) \in \Gamma(T)$ depending on a continuous parameter t so that the corresponding response

$$x(T,t) = W(T)\, x_o + \sum_{j=1}^{T} W(T)\, W^{-1}(j)\, \hat{B}(j-1)\, u(j-1,t) \qquad (3)$$

moves toward $x^* \in K(T,x_o)$ as $t \longrightarrow \infty$ for fixed $0 \leq T < T^*$. To find this path corrections to u are computed from

$$\frac{du}{dt}(j-1,t) = g(x(T,t))$$

for $j = 1,2,\ldots,T$, where $g(x(T,t))$ has yet to be found.

Along a solution curve $x(T,t)$,

$$\frac{dV(x(T,t))}{dt} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial t}$$

$$= 2H\ x(T,t)\cdot\frac{\partial x}{\partial t}(T,t)$$

$$= 2H\ x(T,t)\cdot \sum_{j=1}^{T} h(i)\ \dot{u}(j-1,t).$$

Here $h(j) = W(T)\ W^{-1}(j)\ \hat{B}(j-1)$. Considering only one control variable u, i.e., m = 1, the function $g(x(T,t))$ is defined as follows:

$$\dot{u}(j-1,t) = -\beta(j)\ h(j)\ Hx(T,t) \tag{4}$$

if

$$u(t) = [u(0,t),\ u\ (1,t)\ \ldots\ u(T-1,t)] \in (\Gamma(T) - \partial\Gamma(T)),$$

$$(\text{constant} = \beta(j) > 0)$$

and

$$\dot{u}(j-1,t) = -\tilde{K}(j)\beta(j)h(j)H\ x(T,t) \tag{5}$$

if $u(t) \in \partial\Gamma(T)$, where $0 \leq \tilde{K}(j) \leq 1$ and $\tilde{K}$ with norm $\|\tilde{K}\| = \sum_{j=1}^{T} |\tilde{K}(j)|$ is the largest vector such that the vector $\dot{u}$ based at $u(t) \in \partial\Gamma(T)$ is directed into $\Gamma(T)$. That is, if $u(t)$ lies in the hyperplanes $a^i\cdot u + b^i = 0$ for $i = 1,2,\ldots\nu$ of the defining hyperplanes of $\Gamma(T)$ then along the solution curves $u(t)$, it is required that

$$\frac{d(a^i\cdot u(t) + b^i)}{dt} = a^i\cdot\dot{u}(t) \leq 0.$$

To find this a linear programming problem has to be solved or the method of gradient projection as discussed in reference 1 must be used. The linear programming problem involves finding $\tilde{K}$ for equation (5) such that $\|\tilde{K}\|$ is maximum for $\tilde{K}$ satisfying the constraints:

$0 \leq \tilde{K}(j) \leq 1$, and $a^i \cdot \dot{u}(t) = \tilde{K} \cdot c^i \leq 0$; for $i = 1,2,\ldots\nu$.
To use the gradient projection scheme a matrix (equation 4.12 of reference 1) which is size T x T must be inverted, where T is the dimension of the space u (for one control variable defined on T segments).

Since

$$\dot{V} = -2 \sum_{j=1}^{T} (H\, x(T,t)\cdot h(j))^2 \, \beta(j)\, \tilde{K}(j) \leq 0$$

with $\beta(j) > 0$, $0 \leq \tilde{K}(j)$ 1, it is not hard to prove convergence of $V(x)$ to $V(x^*)$ as $t \longrightarrow \infty$; using the method for the unconstrained case. Unfortunately it requires time to calculate $\tilde{K}$ and therefore the above computation for correction of the control must be discretized. Corrections of the control can be computed from the recurrence equation

$$u^{(i+1)}(j-1) = u^{(i)}(j-1) - k\, \tilde{K}(j)\beta(j)h(j)\, Hx^{(i)}(T,0)$$

relating the value of u at i+1 to the value of u at i for $k > 0$ and $i = 1,2,\ldots$ One additional problem is now encountered in that the step size must be such that the corrected value of the control remains in the restraint set. The step size can be determined by selecting a value of k and then checking the constraints. If they are not satisfied k could be halved, and so on. Rosen proves convergence for this procedure in reference 1, page 193.

A similar analysis is possible for two or more control variables.

Although the above schedule of computations can be carried out using high speed computers to find the optimum control sequence for a given initial state, it does not appear possible to do the calculation on line as can be done for the unbounded phase coordinate case.

CONCLUSIONS

The problem of optimum control of discrete systems subject to phase variables, constraints and constraints on the control variables has been reduced to one involving constraints on the control variables only. An algorithm equation 4, has been developed for determining the control of the new constraint region which provides a minimum to the quadratic cost function. The computation does not appear to be practical for on line solutions.

REFERENCE

1. Rosen, J. B., "The Gradient Projection Method for Nonlinear Programming", Part I and II, J. Soc. for Indust. and Appl. Math., Vol. 8, No. 1, 1960 and Vol. 9, No. 4, 1961.